

Grid Minors in Damaged Grids

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Abstract. We prove upper and lower bounds on the size of the largest square grid graph that is a subgraph, minor, or shallow minor of a graph in the form of a larger square grid from which a specified number of vertices have been deleted. Our bounds are tight to within constant factors. We also provide less-tight bounds on analogous problems for higher-dimensional grids.

1 Introduction

If an $n \times n$ grid graph is damaged by the removal of m of its vertices, how large a grid can be guaranteed to exist in the remaining graph? This is a classical question from distributed computing, where the grid graph is assumed to represent the processors and communication links of an ideal distributed computing system, and one would like to be able to run the system as if it were in its ideal state even when some of its processors have become faulty. An answer was provided by Cole, Maggs, and Sitaraman [6], who showed that a faulty system can simulate the ideal system as long as the number of faults is at most $n^{1-\epsilon}$ for some $\epsilon > 0$. However, their simulation does not find a grid-like subgraph of the remaining undamaged part of the initial grid; it involves loading some processors and links with the work that in the original grid would have gone to a multiple processors and links, and it has $\Theta(n)$ startup time making it efficient only for simulations with high running time. Indeed, prior research had already shown that, if an ideal grid is to be emulated by a damaged grid of the same size by directly embedding the ideal grid into the damaged grid (with constant load, dilation, and congestion) then only a constant number of faults can be tolerated [12].

In recent work on approximation algorithms for the minimum genus of a graph embedding, Chekuri and Sidiropoulos asked a similar question about grids in damaged grids, but with a more graph-theoretic formulation [5]. One step of their genus approximation involves finding a large square grid graph as a *minor* of a damaged grid; a minor of a graph G is a graph H that can be formed from G by deleting edges and vertices and by contracting edges. Chekuri and Sidiropoulos observed that an $n \times n$ grid graph with m damaged vertices always has an undamaged grid *subgraph* of size $O(\frac{n}{\sqrt{m}}) \times O(\frac{n}{\sqrt{m}})$, a result that was sufficient to show that the optimal genus g could be approximated to within an approximation ratio of $g^{O(1)} \log^{O(1)} n$. Chekuri's and Sidiropoulos's bound on grid subgraph size turns out to be tight to within a constant factor. However,

the set of minors that can be formed from a graph may be much larger than its set of subgraphs, so it is reasonable to hope that one can find grid minors that are larger than this bound by a non-constant factor.

Questions about the size of grid minors are central to Robertson and Seymour's work on the theory of graph minors, and notoriously difficult. Following earlier analogous results for infinite graphs [?, ?], Robertson and Seymour showed the existence of a non-constant function f such that every graph of treewidth t has a grid minor of size $f(t) \times f(t)$ [1, 10, 16, 18, 20]. However, although better results are known for special classes of graphs [8, 9, 18] or other structures than grids [17], the growth rate of f is not known in general, and the known upper and lower bounds on its growth rate are far apart. So it may be a bit of a surprise that we can determine the maximum size of a grid minor that we can guarantee to exist in a damaged grid much more precisely, to within a constant factor: it is $\Theta(\min\{n, n^2/m\})$. In particular, if cn vertices are deleted from an $n \times n$ grid (for any constant $c > 0$), we can still find a grid minor of linear size. This result leads to an improvement in the exponent of the $g^{O(1)}$ term in the genus approximation ratio of Chekuri and Sidiropoulos.¹

In this paper we prove this result on the sizes of grid minors in damaged grids, and we generalize it in two ways, to higher dimensional grids (both grids of bounded dimension and unbounded side length, and hypercubes of unbounded dimension) and to *shallow minors*, a formulation of graph minors that incorporates the distributed computing concept of dilation into graph minor theory and unifies the theories of minors and subgraphs [13, 15, 21, 22]. Our results are constructive, in the sense that they lead directly to simple and practical algorithms for finding grid minors in damaged grids, avoiding the high constant factors of many results in graph minor theory. In Section 7 we provide another algorithmic application of these results, to the theory of *bidimensionality*, a characterization of certain graph parameters for which efficient fixed-parameter tractable algorithms and efficient polynomial time approximation schemes can be found [7]. In many cases, the fact that damaged grids have large grid minors can be used to derive new bidimensional parameters from known ones.

2 Planar grid subgraphs

For completeness, we briefly repeat the argument of Chekuri and Sidiropoulos showing that every $n \times n$ grid with m damaged vertices has an undamaged grid of size $O(\frac{n}{\sqrt{m}}) \times O(\frac{n}{\sqrt{m}})$, and that this is tight.

Theorem 1 (Chekuri and Sidiropoulos). *If G is an $n \times n$ grid graph, and D is a set of m vertices in G , then G has a grid of size $k \times k$ that is disjoint from D , where $k = \left\lfloor \frac{n}{\sqrt{m+1}} \right\rfloor$.*

Proof. Partition G into $\lceil \sqrt{m+1} \rceil \times \lceil \sqrt{m+1} \rceil > m$ smaller grids, of size $k \times k$ (possibly with some rows and columns left over). Because there are more than m of these smaller grids, at least one has to be disjoint from D . \square

¹ C. Chekuri and A. Sidiropoulos, personal communications.

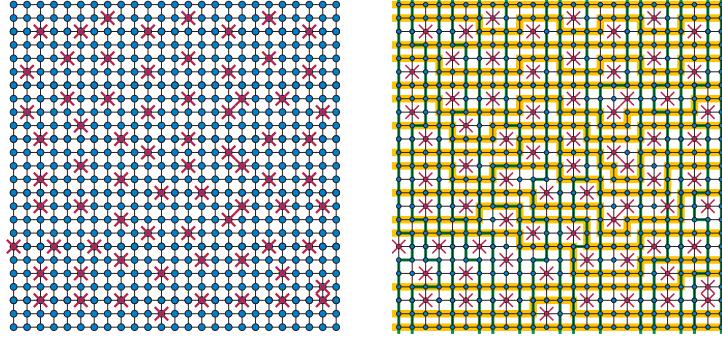


Fig. 1. Left: A 25×25 grid with 72 damaged vertices. There are no undamaged 3×3 grid subgraphs, but many undamaged 2×2 subgraphs. Right: Sets of disjoint paths from one side of the same damaged grid to the other, avoiding the damaged vertices and forming a 15×15 grid minor.

For instance, Figure 1(left) has $n = 25$ and $m = 72$. For these numbers, $\lceil \sqrt{m+1} \rceil = 9$ and $k = 2$. For this example, the bound of Theorem 1 is tight: there are no 3×3 undamaged subgrids, but if we partition it into 144 small 2×2 subgrids (with one row and column left over) then many of them must be undamaged.

A matching upper bound on the size of a grid subgraph is also possible. If we place the vertices of D at positions whose coordinates are both congruent to -1 modulo $k+1$, in a coordinate system for which one of the grid corners is $(0,0)$, then the total number of vertices placed is $\lfloor n/(k+1) \rfloor^2$, and the largest remaining square grid subgraph has size $k \times k$. The inequality $\lfloor n/(k+1) \rfloor^2 \leq m$ has as its maximal solution the same choice of k as in Theorem 1, $k = \left\lfloor \frac{n}{\lceil \sqrt{m+1} \rceil} \right\rfloor$.

Our bound for grid minors will use this same basic idea as Theorem 1, of partitioning into smaller grids in order to make the damage in at least one subgrid sparser than in the original grid, but will find a subgrid that is only lightly damaged rather than one that is not damaged at all.

3 Planar grid minors

When forming a grid minor, rather than a grid subgraph, we may tolerate a greater amount of damage to the original grid, because there are more ways of forming minors than subgraphs. One very versatile way of forming grid minors is by finding many disjoint paths across the grid, as shown in Figure 1(right).

Lemma 1. *Suppose that an $n \times n$ grid is damaged by the deletion of a set of m vertices, but that we can find a collection of k vertex-disjoint paths extending horizontally from one vertical side of the grid to the other, and another collection of k vertex-disjoint paths extending vertically from one horizontal side of the*

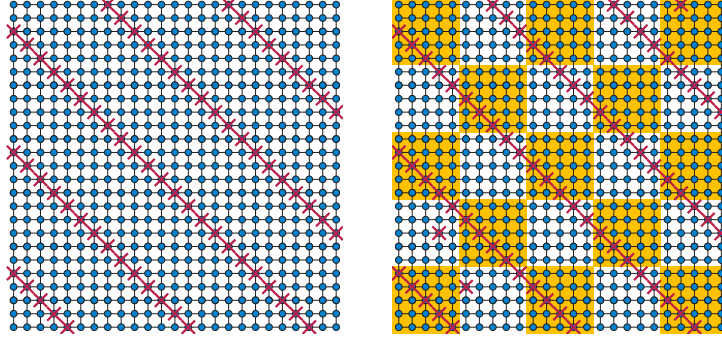


Fig. 2. Left: A 25×25 grid with 69 damaged vertices in the pattern given by the proof of Theorem 4. As shown in that theorem, its largest square grid minor is 4×4 . Right: Even with 72 damaged vertices, partitioning the grid into 25 5×5 subgrids leads to a subgrid with at most two damaged vertices, which necessarily has a 3×3 grid minor.

grid to the other. Suppose also that each horizontal-vertical pair has a connected intersection. Then the damaged grid contains a $k \times k$ grid minor.

Proof. Delete any edges and vertices of the grid that do not belong to the paths. Contract each intersection between a horizontal and vertical path to form each of the grid vertices, and contract the portion of each path between two of these intersections to form the grid edges. A version of the Jordan curve theorem ensures that the intersections on each path lie in the correct order, so the contracted graph is the desired grid. \square

Figure 1(right) shows the paths of the lemma as yellow and green; in this example, there are $k = 15$ paths of each type, so the lemma gives us a 15×15 grid minor, much larger than the largest undamaged grid subgraph.

Theorem 2. Suppose that an $n \times n$ grid is damaged by the deletion of a set D of m vertices. Then the remaining graph has a grid minor of size $k \times k$, where $k = \max\{n - m, n^2/4m - O(1)\} = \Theta(\min\{n, n^2/m\})$.

Proof. To achieve $k = n - m$, observe that the rows and columns of the grid that are disjoint from D form two sets of at least $n - m$ disjoint paths, as required by Lemma 1.

To achieve $k = n^2/4m - O(1)$, partition the given grid into approximately $4(m/n)^2$ subgrids of size approximately $n^2/(2m) \times n^2/(2m)$. The average number of damaged vertices per subgrid is $\frac{m}{4(m/n)^2} = n^2/4m$. There exists at least one subgrid whose number of damaged vertices is at most this average, which is half of the side length of the subgrid. Within this subgrid we may apply the $n' - m'$ bound (where n' is the side length of the subgrid and m' is its number of damaged vertices) giving us a minor of size approximately $n^2/(2m) - n^2/4m = n^2/4m$.

Finally, we observe that if $m < n/2$ then $\max\{n - m, n^2/4m - O(1)\} = \Theta(n)$ while if $m \geq n/2$ then $\max\{n - m, n^2/4m - O(1)\} = \Theta(n^2/m)$, so our bound achieves the stated asymptotics. \square

Proof. We choose D to be a set of m vertices extending diagonally from one corner of the grid towards the center, as shown in Figure 3(bottom). We claim that the resulting damaged grid has pathwidth exactly $n - m$, and so cannot contain a square grid minor of larger than the stated size. In the case when $m = \lfloor n/2 \rfloor$, a path decomposition with width $n - m$ may be found by putting the center vertex into all bags (in the case n is odd) and otherwise making the ordering of the first bag containing each vertex be the radial sorted ordering of these vertices around the center (starting clockwise of the damaged diagonal and breaking ties arbitrarily), as shown in the figure. If m is smaller than $\lfloor n/2 \rfloor$, a decomposition with pathwidth $n - m$ may be obtained by using this same pattern for all vertices that do not lie on the diagonal line segment between the damaged corner and the center, and by including the undamaged vertices that do lie on this line segment into all bags. \square

We also obtain an upper bound for larger values of m that differs from our lower bound only by a factor of two:

Theorem 4. *For $m > n/2$, there exists an $n \times n$ grid and a set D of m damaged vertices in the grid, such that the largest square grid minor using only undamaged vertices has size $k \times k$, with $k = \left\lceil \frac{n^2}{2m} - \frac{1}{2} \right\rceil$.*

Proof. Number the negatively-sloped diagonals of the grid from 1 to $2n - 1$, choose a number r , and damage all the vertices that belong to diagonals numbered r modulo $2k + 1$. At least one of the residue classes of diagonals modulo $2k + 1$ must have at most $\frac{n^2}{2k+1} \leq m$ damaged vertices in it. The pathwidth of the remaining sets of $2k$ contiguous diagonals is at most k , as may be shown by a path decomposition that sorts the vertices by the linear combination $x - y$ of their Cartesian coordinates, breaking ties arbitrarily. Therefore, the largest square grid minor in the remaining graph can have size at most $k \times k$. \square

Figure 2(left) shows an example of this diagonal damage pattern for $n = 25$ and $k = 4$. The pattern of damage shown in the figure reduces the pathwidth of the remaining graph to 4, so its largest undamaged square grid minor has size 4×4 . Theorem 4 shows that damaging at most 70 damaged vertices in a 25×25 grid can block the existence of an undamaged 5×5 grid minor, but as the figure shows a careful choice of r leads to only 69 damaged vertices.

4 Shallow minors

The *radius* of a graph G is smallest number r such that all vertices of G are within distance r of one of its vertices, called its *center*. That is, the radius is

$$\min_{v \in V(G)} \max_{w \in V(G)} \text{distance}_G(v, w).$$

If V_i are disjoint sets of vertices, each of which induces a subgraph of radius at most λ , then the minor of G formed by contracting each set V_i to a single vertex

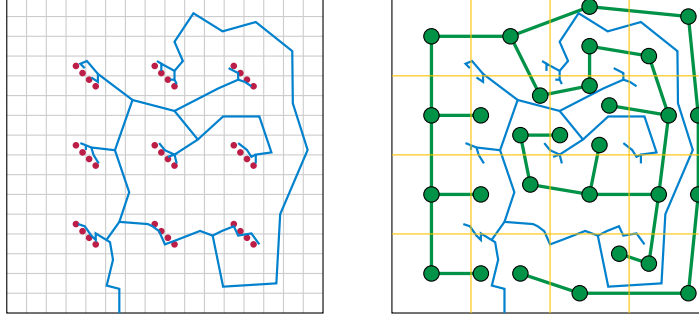


Fig. 4. Schematic view of upper bound construction for shallow minors. Left: The damaged vertices form a collection of diagonal segments of length $2\lambda + 1$, evenly spaced throughout the grid (red). The tree T_H in the dual graph, connecting faces near these segments and avoiding shallow minor H , is shown in blue. Right: Partitioning the grid into subgrids with corners at the centers of the damaged segments (yellow), and the tree K representing the connected components of subgrids after T_H is deleted (green).

is called a *shallow minor* of G , at depth λ . Thus, a shallow minor of unbounded depth (or of depth at least $n - 1$) is just a minor, while a shallow minor of depth 0 is exactly a subgraph. The results of the previous two sections, on grid subgraphs and grid minors, naturally raise the question of how large a square grid can be found as a shallow minor of a given depth λ in a damaged grid.

Theorem 5. *Let D be a set of m vertices in an $n \times n$ grid, and let $\lambda \geq 1$ be given. Then there exists a shallow square grid minor at depth λ in the grid, disjoint from D , of size $k \times k$, where $k = \Omega(\min\{n, n\sqrt{\frac{\lambda}{m}}, \frac{n^2}{m}\})$. For every n , m , and λ , it is possible to choose D in such a way that the largest shallow grid minor of the grid has side length $O(\min\{n, n\sqrt{\frac{\lambda}{m}}, \frac{n^2}{m}\})$. The constants in the O - and Ω -notation used here do not depend on λ , m , or n .*

Proof. To prove existence of a large shallow grid minor we consider the following cases. If $m \leq 2\lambda$, then the grid minor formed by the undamaged rows and columns of the grid has size $(n - m) \times (n - m)$, and depth at most λ . If $m > 2\lambda$ and $\lambda \leq cn^2/m$ for a sufficiently large constant c , we may partition the grid into at least $m/2\lambda + 1$ subgrids of side length $\Theta(n\sqrt{\lambda/m})$; one of these subgrids has at most 2λ damaged vertices in it, and its undamaged rows and columns form a shallow grid minor. The assumption that $\lambda \leq cn^2/m$ implies that $\lambda \leq n\sqrt{cd/m}$ and hence that when we subtract the 2λ damaged rows and columns from the $\Theta(n\sqrt{\lambda/m})$ side length of the subgrid, the number of remaining undamaged rows and columns will still be $\Omega(n\sqrt{\lambda/m})$. In the remaining case, $\lambda = \Omega(n^2/m)$ and we may apply the solution for arbitrary grid minors, as it will automatically provide a minor that is large enough and shallow enough.

The construction of a set D that blocks the existence of large shallow grid minors is trivial when $\min\{n, n\sqrt{\lambda/m}, n^2/m\} = n$, and follows from Theorem 4 when the minimum is n^2/m . The remaining case combines ideas of planar embedding, treewidth, and interdigitating trees. Choose D to be the vertices in at most $\lfloor m/(2\lambda + 1) \rfloor$ diagonal line segments, each containing $2\lambda + 1$ vertices, with these segments evenly spaced across the grid G as shown in Figure 4(left). Any minor of G inherits a planar embedding from the embedding of G , but a square grid minor has a unique planar embedding up to the choice of its outer face, in which all faces other than the outer face are 4-cycles. None of these 4-cycles can completely surround one of the diagonal line segments in D without exceeding the depth constraint, so all of the diagonal line segments of D must lie within the outer face of any shallow grid minor. By similar reasoning the outer face of G itself must also lie within the outer face of any shallow grid minor. It follows that we can find a connected tree T_H in the dual graph of G , spanning the outer face of G and all of the faces of G that are adjacent to vertices of D , with the property that T_H is disjoint from the edges of G that have both endpoints in sets V_i . Each shallow grid minor H of $G \setminus D$ is also a minor of the graph $G \setminus T_H$ formed by deleting the edges of G that are dual to edges in T_H . The tree T_H is illustrated in Figure 4(left).

No matter how such a tree T_H is formed, the resulting graph $G \setminus T_H$ will have treewidth $O(n\sqrt{\lambda/m})$. For, consider the partition of G into subgrids with corners at the centers of the diagonal damaged segments, and form a graph K in which the vertices are connected components of the intersection of subgrids with $G \setminus T_H$ and the edges are adjacent pairs of components (Figure 4(right)). K is a tree (it cannot contain a cycle, for such a cycle would surround one of the damaged diagonal segments preventing it from being attached to the rest of T_H), and hence has treewidth one. A tree-decomposition of $G \setminus T_H$ of width $O(n\sqrt{\lambda/m})$ may be obtained by using the decompositions of the subgrid components to expand the tree-decomposition of K .

We have seen that any shallow square grid minor H of G is a minor of $G \setminus T_H$, and that the treewidth of $G \setminus T_H$ and of its grid minors is $O(n\sqrt{\lambda/m})$. But the treewidth of a grid is its side length [2], so the largest shallow square grid minor in $G \setminus T_H$ must have side length $O(n\sqrt{\lambda/m})$. \square

5 Grids of bounded dimension

Cubical grids of dimension higher than two (i.e. Cartesian products of equal-length paths) are less central than planar grids to the theory of graph minors, but there has been some past study of properties of these graphs related to minors [3, 4, 14]. As we show in this section, many of our results for planar grids extend directly to grids of higher dimension. Throughout this section, when we use O -notation, we treat the dimension d as a fixed constant (suppressing the dependence on D of the constant factors in the O -notation).

Theorem 6. *Let G be a d -dimensional cubical grid of side length n with a set D of m damaged vertices. Then G has a d -dimensional cubical grid subgraph of*

side length $\Omega(n/m^{1/d})$ disjoint from D . There exist sets D with $|D| = m$ for which every d -dimensional cubical grid subgraph disjoint from D has side length $O(n/m^{1/d})$.

Proof. As in Theorem 1, the lower bound partitions the grid into more than m subgrids of the given side length, at least one of which must be undamaged. The upper bound places the vertices of D on points with Cartesian coordinates that are all -1 modulo $k + 1$, for k chosen to make $|D| \leq m$. \square

Theorem 7. *Let G be a d -dimensional cubical grid of side length n with a set D of m damaged vertices. Then G has a d -dimensional cubical grid minor of side length $\Omega(\min\{n, (n^d/m)^{1/(d-1)}\})$ disjoint from D . There exist sets D with $|D| = m$ for which every d -dimensional cubical grid minor disjoint from D has side length $O(n^d/m)$.*

Proof. For the lower bound, a grid with $m < n$ has a grid minor of side length $n - m$, formed by the intersection pattern of its undamaged $(d - 1)$ -dimensional axis-parallel hyperplanes. The result follows by partitioning the grid into subgrids whose average number of damaged vertices is proportional to their side length, and selecting a subgrid whose number of damaged vertices is at most average.

The upper bound places the vertices of D on evenly spaced $(d - 1)$ -dimensional axis-parallel hyperplanes within the grid, partitioning it into subgrids of the given size that are disconnected from each other. \square

Unlike the two-dimensional case, the upper and lower bounds of Theorem 7 do not match, essentially because in higher dimensions the side length of a cube is no longer proportional to its surface measure.

Theorem 8. *Let G be a d -dimensional cubical grid of side length n with a set D of m damaged vertices, and let $\lambda \geq 1$ be given. Then G has a d -dimensional cubical grid shallow minor of depth λ and side length*

$$\Omega(\min\{n, n(\lambda/m)^{1/d}, (n^d/m)^{1/(d-1)}\})$$

disjoint from D .

Proof. Partition the grid into subgrids within one of which there must be at most 2λ damaged vertices, and then find a minor using the remaining undamaged axis-parallel hyperplanes, as in Theorem 5. \square

Theorem 8 generalizes the lower bound of Theorem 5 to higher dimensions, but, we do not know how to generalize the corresponding upper bound.

6 Hypercubes

For grids of unbounded dimension, our knowledge is even more limited, but we can still prove some results. We confine our attention to the case $n = 2$ of hypercube graphs; we let $N = 2^d$ denote the number of vertices in such a graph.

Theorem 9. *For every hypercube graph with N vertices, and every set D of m damaged vertices, there is a hypercube subgraph disjoint from D containing at least $N/2m$ vertices.*

Proof. Let p be the smallest power of two that is strictly larger than m , and (by dividing the vertices into equivalence classes according to the value of their first $\log_2 p$ coordinates) partition the hypercube into p smaller hypercubes of N/p vertices each. At least one of these smaller hypercubes must be disjoint from D . \square

Theorem 10. *For every hypercube graph with N vertices, and every m , there exists a set D with $|D| = m$ such that the largest hypercube subgraph disjoint from D has $O((N \log N)/m)$ vertices.*

Proof. We apply the probabilistic method to prove the existence of D . If D is chosen uniformly at random, among all m -vertex sets, then the probability of a single vertex not belonging to D is $1 - m/N$ and the probability of a subcube of size kN/m being disjoint from D is approximately $\exp(-k)$. There are $N^{\log_2 3}$ subcubes of the hypercube (obtained by specifying, for each coordinate, whether it is fixed to 0 or 1 or free to vary), and when k is a sufficiently large multiple of $\log N$, $N^{\log_2 3} \ll \exp(k)$. For such k , the expected number of subcubes of size kN/m that are disjoint from D is strictly less than one, but this number is the expected value of a non-negative integer random variable (the number of subcubes disjoint from D for a particular random choice of D) so there must exist a choice of D that makes the number of disjoint subcubes zero. \square

We say that coordinate i is a *bad coordinate* for a set D of damaged vertices of a hypercube if there are two vertices in D at distance at most two from each other, such that the one or two coordinates on which these vertices differ include i . If i is not a bad coordinate, then contracting all edges of the hypercube between pairs of vertices that differ in coordinate i results in an undamaged hypercube minor of dimension smaller by one, because every contracted vertex is formed from at least one undamaged vertex of the original hypercube and every edge of the contracted hypercube corresponds to at least one edge between two undamaged vertices of the original hypercube.

Lemma 2. *For every hypercube and every set D of m vertices, there are at most $2m - 2$ bad coordinates.*

Proof. Draw a graph F on D , connecting a subset of the pairs of vertices at distance two from each other, with the subset chosen to be minimal with the property that every bad coordinate i is represented by an edge in F between two vertices that differ in coordinate i . Then F must be acyclic, for if an edge e belonged to a cycle then the bad coordinates represented by e would also each be represented by at least one other edge of the cycle. Therefore, F is a forest, with at most $m - 1$ edges. Each bad coordinate is covered by an edge in F , and every edge covers at most two bad coordinates, so the number of bad coordinates is at most $2m - 2$. \square

An example showing Lemma 2 to be tight may be constructed by letting D consist of the origin and of $m - 1$ points at distance two from it, no two sharing the same nonzero coordinate.

Theorem 11. *For every hypercube graph with N vertices, and every set D of m damaged vertices, there is a hypercube shallow minor of depth 1 disjoint from D and containing $\Omega(\min\{N, (N/m) \log(N/m)\})$ vertices.*

Proof. If $m \leq \frac{1}{2} \log_2 N$, by Lemma 2 there is a coordinate that is not bad, and contracting that coordinate produces a hypercube minor with $N/2$ vertices. Otherwise, let p be the smallest power of two greater than $2m/\log_2(N/2m)$, partition into p smaller hypercubes, and choose one in which the number of damaged vertices is at most $\frac{1}{2} \log_2(N/2m)$. The number of vertices in this hypercube is large enough that there necessarily exists a coordinate that is not bad, and again contracting that coordinate produces the desired minor. \square

For example, with $N = 8$ and $m = 2$, choosing D to be the two opposite corners of a cube reduces the number of vertices in the largest hypercube subgraph to be only two; however, no matter how D is chosen, at most two of the three coordinates may be bad, so there is always a way to contract one coordinate and produce a square minor with four vertices.

7 Bidimensionality

A *graph parameter* is described by a function from graphs to the natural numbers; for instance, the most frequently used graph parameters in algorithm analysis are the numbers of vertices and edges. A graph parameter is said to be *minor-monotone* if contracting or deleting an edge can only cause the parameter to decrease or remain unchanged. A minor-monotone graph parameter is said to be *bidimensional* (or more specifically *minor-bidimensional*) if its value on an $n \times n$ square grid is $\Omega(n^2)$; that is, it must be at least proportional to the number of vertices in the grid [7].

Because of the close connections between grid minors and treewidth, bidimensional parameters have a highly developed algorithmic theory, centered around the fact that, for every graph, either there is a large grid (and the parameter has a high value) or the graph has low treewidth (and efficient algorithms for graphs of low treewidth may be applied). In particular, bidimensional parameters may be computed exactly by a fixed-parameter tractable algorithm (parameterized by the parameter that we are trying to compute) whose dependence on its parameter is subexponential, and many of these problems (with certain additional properties) may also be approximated efficiently [7]. Thus, it becomes of great interest in algorithms research to identify natural parameters that are bidimensional.

Let p be any graph parameter, and define another parameter $\Delta_{p,k}$, where $\Delta_{p,k}(G)$ is the minimum number of vertices that need to be deleted from graph G to reduce it to a subgraph G' with parameter value $p(G') \leq k$. In many cases we will take $k = 0$ or $k = O(1)$ and omit it from the notation.

Theorem 12. *If p is minor-bidimensional, then so is $\Delta_{p,k}$.*

Proof. If p is minor-monotone, then so is $\Delta_{p,k}$. For, if D is any set of vertices in G whose deletion reduces p to k then in any graph formed by edge deletion or contraction from G we can achieve at least the same reduction by deleting the vertices formed from D by the contraction operations.

Additionally, if p is minor-bidimensional, then $\Delta_{p,k}$ is $\Omega(n^2)$ on the $n \times n$ grid graph. For, if p is nonconstant on grids (as is true for every minor-bidimensional parameter, by definition) then by Theorem 2 it requires $\Omega(n^2)$ vertex deletions to reduce the $n \times n$ grid to a graph in which all grid minors have bounded size, on which p might possibly be reduced to a constant. \square

Unfortunately repeating the Δ operator a second time does not produce a new parameter: $\Delta_{\Delta_p} = \Delta_p$. However, this theory does provide an immediate explanation for the bidimensionality of several well-known and well-studied graph parameters: for instance the vertex cover number is the result of applying the Δ operator to the number of edges, and the feedback vertex set number is the result of applying the Δ operator to the cyclomatic number.

By combining the Δ operator with other bidimensional parameters we may obtain additional parameters that do not appear to have been studied previously. If ℓ is the length of the longest path in a graph, for instance, then $\Delta_{\ell,0}$ is again the vertex cover number, but $\Delta_{\ell,k}$ for nonzero values of k appears novel and interesting.

The theory of bidimensionality is most usefully applied to parameters that are not themselves polynomial-time computable, but the Δ operator may be of interest when applied even to parameters that are easy to compute. An example was given above, in which applying it to the edge count produced the vertex cover number. Another example is given by the *branch count* [11], which in a connected graph is the sum, over the vertices v , of $\max\{0, \deg(v) - 2\}$, and in a disconnected graph is the maximum value of this sum in a single connected component. The domination number of a 3-regular graph may be given as $\Delta_{p,0}$ for this parameter, allowing minor-bidimensionality to be applied in this case – in general, the domination number is contraction-bidimensional but not minor-bidimensional – while again applying $\Delta_{p,k}$ for nonzero values of k gives different parameters.

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